"Gothic" characters. That the spacing into groups of threes or into groups of fives laterally and vertically has an important bearing upon legibility and quick understanding is highly probable; that the use of white spaces in the place of black rulings works great change is undoubted. Which of these, and what other features are to be preferred in tables, is of great consequence. These questions can only be answered after very exhaustive experiments of no mean order. A competent psychologist and a competent mathematician working jointly would make the best combination for attempting such a great task.

5. **Forthcoming Articles.** We are glad to announce that short articles dealing with very specific topics are already promised. Among others we may mention: an article on the teaching of solid angles; one on the desirability of accurate space drawings; one on methods of interpolation in a table of logarithms of the trigonometric functions for small angles; one on the elements of the theory of insurance. It is hoped that these and others equally definite in topic will not be long delayed. In this connection see the note at the bottom of page 140 of this issue.

E. R. HEDRICK, *for the editors.*

HISTORY OF THE EXPONENTIAL AND LOGARITHMIC CONCEPTS.

By FLORIAN CAJORI, Colorado College.

IV. FROM EULER TO WESSEL AND ARGAND. 1749–about 1800.

A HALF CENTURY OF BARREN DISCUSSION ON LOGARITHMS.

In an essay *Sur les logarithmes des quantités negatives*[1] D'Alembert refers to his correspondence of 1747 and 1748 with Euler on this subject and states that he had read Euler's paper of 1749, but felt still that the question was not settled. D'Alembert proceeds to advance arguments of metaphysical, analytical and geometrical nature which shrouded the subject into denser haze and helped to prolong the controversy to the end of the century. Defining logarithms by the aid of two progressions, as did Napier, D'Alembert asserts that the logarithms of negative numbers are not imaginary, but real, or rather, that they may be represented at will as real or imaginary, since everything hinges on the choice of the system of logarithms. As his metaphysical argument against Euler's conclusion he gives it as improbable that the logarithmic curve $y = e^x$ passes from $x = \infty$ to imaginary values. As a geometric reason he asserts that all curves for which $a^n dy/y^n = dx$, n being odd, have two branches which are symmetrical with respect to the X-axis and yielding two values, y and $-y$ for one and the same value of x. He claims his proof to be general, hence true for $n = 1$.

[1] *Opuscules mathématiques*, T. I, Paris, 1761, pp. 180–209.

This conclusion, says Karsten,[1] rests upon an error in sign, so simple that he doubts whether D'Alembert was really in earnest in his argument. The disagreement between D'Alembert and Karsten was due to a lack of a clear and consistent definition of negative areas, and to the use of infinite areas involving the indeterminate form $\infty - \infty$. No doubt D'Alembert and others had a right to assume two branches to the logarithmic curve, if they wished; the whole question is one of assumption. If we agree that $y = e^x$ shall be so interpreted that for every fractional value of x with an even denominator, both real roots shall be taken, then there are two branches, of course; if we agree to restrict ourselves to the positive root only, then there is naturally only one branch. But what most mathematicians of that time failed to understand, was Euler's remark that the question of one branch or two branches has in fact nothing to do with the question as to the nature of logarithms of negative numbers. The former is a question in evolution, the second is a question in logarithmation; the former has reference to one of the two inversions of involution, the latter to the other inversion. In his paper of 1749 Euler had explicitly confined himself to the system with the positive base $e = 2.718 \cdots$ and had concluded that the logarithms of negative numbers are complex. D'Alembert selects a negative base by taking a geometric series $1, -2, 4, -8, \cdots$ and the arithmetic series $0, 1, 2, 3, \cdots$. Thus he obtains certain negative numbers that have real logarithms. When Euler says that $l(+a) = l(-a)$ yields zero for the logarithm of all complex numbers of unit modulus, D'Alembert in reply again changes his system of logarithms to one in which the arithmetic series is $0, 0, 0, \cdots$ and reminds the reader that any geometric series whatever can be associated with this. For one and the same system of logarithms D'Alembert selects the two series

$$\begin{array}{cccccccccccccc} \cdots & 2n & n & 0 & -n & -2n & \cdots & \infty & \cdots & -2n & -n & 0 & n & 2n \cdots \\ \cdots & -b^2 & -b & -1 & 1/b & 1/b^2 & \cdots & 0 & \cdots & 1/b^2 & 1/b & 1 & b & b^2 \cdots \end{array}$$

in which the base is arbitrarily altered in passing from -1 to $1/b$; this is done to show that $lb^m = l(-b^m)$. What other mathematician of renown ever changed his base so often in defence of a mathematical theory!

Though not published until 1761, this paper of D'Alembert was written some years previous, certainly before the publication in 1759 of a paper, *Reflexions sur les quantitées imaginaires*[2] by Daviet de Foncenex. This paper is the more interesting because its author, a pupil of Lagrange, remarks that Lagrange had communicated to him his views on the logarithms of negative numbers. Lagrange himself never published anything on this subject. De Foncenex gives an elementary proof for $\log(\cos\varphi + i \sin\varphi) = \varphi\sqrt{-1}$, then writes $\cos\varphi + \sqrt{-1}\sin\varphi = a + b\sqrt{-1}$, and takes for φ, $\varphi + 2\lambda\pi$. Then $(\varphi + 2\lambda\pi)\sqrt{-1} = \log(a + b\sqrt{-1})$. This cotesian result is derived by de Foncenex with the aid

[1] W. J. G. Karsten, "Von den Logarithmen verneinter Grössen," 1. und 2. Abtheilung, *Abh. Münch Acad.*, V, 1768, S. 55.

[2] *Miscellanea Philosophico-mathematica societatis privatæ Taurinensis*, I, 1759, pp. 113–146.

of the hyperbola, the very curve by which J. Bernoulli I derived results contradictory to this. He says that Bernoulli did not pay proper attention to questions of continuity. In $dz/z = -du/u$, the element of surface dz/z becomes finite while u is passing from $+\infty$ to $-\infty$. He argues that, since the element of surface changes its sign without passing through zero, there is no continuous transition from positive to negative surfaces of the hyperbola; since $z = a^u$ yields $du = dz/z$, there is no continuous passage from the $+$ to the $-$ branch of the logarithmic curve; the branches are real, but isolated and algebraically independent of each other, though transcendentally connected. Apparently in contradiction of his own earlier analytical conclusions, de Foncenex now grants that the logarithms of negative numbers may be considered real, like those of positive numbers.

A reply to de Foncenex was prepared by D'Alembert, under the title, *Supplément au mémoire précédent.*[1] This reply appeared at the same time and in the same volume as his first article. D'Alembert discusses questions of geometry, corrects some mistakes of de Foncenex and clings to some erroneous views of his own.

De Foncenex prepared a reply to D'Alembert's two papers,[2] in which he declares that the arithmetical side of this controversy must be kept distinct from the geometrical. If in $y = e^x$, e is a fixed number, there can be no real logarithms of negative numbers. In the study of the logarithmic curve, we must inquire into the existence of two branches and their relation to each other without using processes of integration. He finds the ordinate β of the evolute of $y = e^x$ to be $(2y^2 + 1) \div y$. Not satisfied with this, he puts $1 + y^2 = u$ and finds $\beta^2 = (4u - 4u^2 - 1) \div (1 - u)$. Hence, he says, there are two values for β or there are two branches of the evolute, both symmetrical with respect to the X-axis. Since the evolute has two branches, the curve itself must have two branches. He now admits D'Alembert's contention that the two branches are algebraically connected. He enters upon the impossible task of harmonizing results by writing $\varphi\sqrt{-1} = \log(\pm \cos\varphi \pm \sin\varphi\sqrt{-1})$, with the direction that the proper signs be chosen in every special case. It is seen that de Foncenex started out in close alignment with the views of Euler, but departed from them considerably through his study of questions of continuity of curves and his false interpretation of the roots of a quadratic equation.

D'Alembert wrote the article "Logarithme" for Diderot's *Encyclopédie ou dictionnaire raisonné,* 9, Paris, 1765. This article was later embodied in the great *Encyclopédie méthodique,* 1785, the mathematical part of which was translated into Italian in 1800. On the subject of logarithms of negative numbers the article voiced the views of J. Bernoulli I. rather than those of Euler. The only other Frenchman who before the closing years of the eighteenth century is known to us as participating in this discussion was Louis Charles Trincano, who con-

[1] *Opuscules mathématiques,* T. I, Paris, 1761, pp. 210–230.

[2] De Foncenex, "Eclaircissemens, etc.," *Mélanges de phil. et de math. de la soc. roy. de Turin pour les années* 1760–1761, pp. 337–344.

tributed a *Mémoire sur les logarithmes des quantités négatives* as an appendix to his father's *Traité complet d'arithmétique,* Paris et Versailles, 1781. The young man says truthfully enough that if one assumes the geometric progression $-2, -4, -8, \cdots$ or $2, -4, 8, \cdots$ or $-2, 4, -8, \cdots$ then some or all negative numbers have real logarithms. According to the first progression, positive numbers would have no real logarithms. In $2, 4, 8, \cdots$ the logarithm of a negative number is *chimérique.*

Notwithstanding the influence of D'Alembert, Euler's theory acquired a firm foothold in France at the close of the century. In 1797–9 appeared in Paris Lacroix's great treatise on the calculus, which presents logarithms unreservedly from the eulerian stand-point.[1]

It is worthy of note that we have not encountered a single research, published in England in this century and touching the topic under discussion. Can this be due to the alienation between English and continental mathematicians which resulted from the Newton-Leibniz controversy on the invention of the calculus?

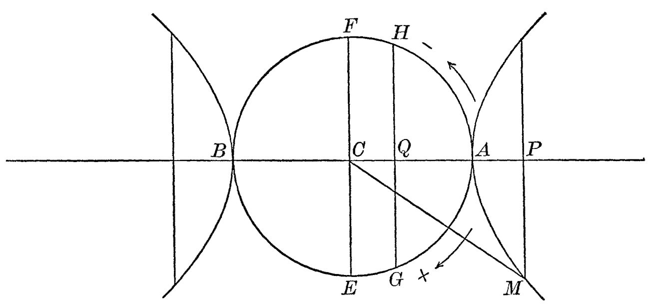
In Germany there appeared between 1750 and 1770 three writers who accepted the eulerian view and were unusually clear in their expositions. We refer to Charles Walmesley, an English Roman catholic prelate who was elected member of the Berlin Academy and contributed a paper to its memoirs, J. A. Segner, professor at Halle, and W. J. G. Karsten, professor at Bützow, later at Halle. Walmesley wrote an article of four pages, *Méthode de trouver les logarithmes de chaque nombre positif, négatif, ou meme impossible,*[2] which contains a brief derivation, by the integral calculus, of the formula $ac\sqrt{-1} = \pm \log (y \pm u\sqrt{-1})$, where ac is an arc of unit-circle, y and u the cosine and sine of the arc. From this formula, and the consideration of the periodicity of trigonometric functions, Walmesley deduces the Eulerian values for $\log 1$, $\log -1$, $\log \pm \sqrt{-1}$, $\log (a + b\sqrt{-1})$. In this article, as in Euler's article of 1749, y and $u\sqrt{-1}$ are tacitly taken along lines perpendicular to each other. To find the logarithms of $a + b\sqrt{-1}$, Walmesley changes it to the form $(a^2 + b^2)^{-\frac{1}{2}}(y + u\sqrt{-1})$; now he says, *il est clair qu'on doit prendre l'arc de cercle dont le sinus est u, le cosinus y,* then if the arc is φ and $\log (a^2 + b^2)^{\frac{1}{2}} = C$, we have for $\log (a + b\sqrt{-1})$ the values $C + \varphi\sqrt{-1}$, $C + (\varphi \pm 2\pi)\sqrt{-1}, \cdots$. We suspect that the diagram attributed to Wessel and Argand was in the *minds* of mathematicians long before it was recorded on *paper.* Walmesley closes with some general remarks disclosing an uncomfortable feeling toward the negative and imaginary. He declares it improper to speak of the ratio between a positive and a negative quantity, since ratio involves only magnitude, not quality. To mix the two ideas is like considering density, weight, etc., in comparing the volumes of two solids.

Segner lays unusual emphasis for that time upon the treatment of imaginaries and is the first to embody Euler's logarithms of complex numbers in a regular school text. In Segner's *Cursus mathematici, Pars* IV, *Halæ Magdeburgigæ,*

[1] S. F. Lacroix, *Traité du calcul dif. et calcul int.*, seconde éd., T. I, 1810, Introduction, §§81, 82.
[2] *Histoire d. l'acad. roy. d. sc. et belles-let.* [*Mémoires*], année 1755, Berlin, 1757, pp. 397–400.

1763, there is a discussion of the roots of unity, treated by De Moivre's formula, followed by a full treatment of the logarithms of complex numbers. Euler's position is here taken fearlessly. The subject is emphasized in *Pars* V (1768) in a section on *De usu imaginariorum.* Segner's treatment is analytical throughout and free from entangling alliance with the then puzzling questions relating to the logarithmic curve and hyperbola.

Karsten published an extensive article of 108 pages[1] in which D'Alembert's errors are brought out with great clearness. But Karsten agrees with D'Alembert in one point, namely that Euler's proof (quoted by us in full on p. 81) of the fundamental theorem, that $\log x$ has an infinity of distinct values, is objectionable in this respect: It conveys the idea that all logarithms belong to positive numbers, since $(1 + \omega)^n$, where ω is infinitesimal and n is infinite, appears to be always positive; the controversy itself relates to negative numbers. As a matter of fact, Euler makes no such restriction; he permits ω to be imaginary. His only error consisted in not giving an explanation sufficiently exhaustive for the needs of the time. Karsten is probably right in claiming that it would have been better, had Euler used in the proof a more elementary definition of a logarithm. Against D'Alembert, Karsten argues convincingly, that D'Alembert simply shows the possibility of systems in which negative numbers have real logarithms; the question at issue being, however, what are the logarithms of negative numbers in the natural system, with the base $e = 2.718 \cdots$? Most interesting in Karsten's paper is a geometric representation of the infinitely-multiple logarithms of a number. We have never seen a reference to this ingenious construction either by contemporaries of Karsten or by historians. It looks very much as if the transactions of academies had been in some cases the safest places for the concealment of scientific articles from the scientific public.



In going over Karsten's construction it must be remembered that he wrote 29 years before Wessel. In the hyperbola $x^2 - y^2 = 1$ and the circle $x^2 + z^2 = 1$, where Karsten takes $y = z\sqrt{-1}$, the "circle is an imaginary part of the hyperbola" and *vice versa.* An arc AG of the circle may be considered as an imaginary

[1] "Abhandlung von den Logarithmen verneinter Grössen," 1. und 2. Abtheilung, *Abh. Münch. Acad.*, V, 1768.

arc of the hyperbola. Each of the four arcs which come together at A or B is a continuation of each of the others, since all are expressed by a common equation. Between each two points M and G exist numberless different arcs: MAG, $MAGBFAG$, and in general, $MAG + 2\lambda\pi$, and also $MAHBG + 2\lambda\pi$, where $\lambda = 0, 1, 2, \cdots$. To any abscissa x there belong therefore not only numberless arcs, but also numberless corresponding sectors. Let one sector of the abscissa x be $\frac{1}{2}\log(x + y) = \sqrt{-1}\cdot\frac{1}{2}$ arc cos $x = \sqrt{-1}$ sect. cos x, the sector being assumed 0 when $x = 1$. If $x > 1$, say $x = CP$, then $\frac{1}{2}\log(x + y)$ gives the sectorial areas corresponding to the arcs $AM \pm \lambda(AGBFA)$, that is, $AM \pm 2\lambda\pi\sqrt{-1}$. Only the sector ACM is real. If $x < 1$, then $x + y$ is imaginary Let $x = CQ$, then $\frac{1}{2}\log(x + y)$ measures the sectorial areas whose arcs are $AG \pm 2\lambda\pi\sqrt{-1}$, or $AFBG \pm 2\lambda\pi\sqrt{-1}$. All these arcs and sectors are imaginary. From this we see that $\frac{1}{2}\log(-1)$ is represented by the imaginary sectors whose arcs are $AEB + \lambda(BFAEB)$, or also $AFB + \lambda(BEAFB)$. These values are all included in the formula $\log(-1) = \pm(2\lambda + 1)\pi\sqrt{-1}$. "Geometry is therefore so little antagonistic to the leibnizian tenet, that the logarithms of negative numbers are impossible, as actually to give it full confirmation."[1] In this representation of logarithms there is established a double correspondence: one between points in the plane, and x and y; the other between points in the plane, and x and z, where $y = z\sqrt{-1}$.

The country in which our controversy was carried on with greatest zest during the second half of the eighteenth century is Italy. We have already referred to de Foncenex's two papers. A few years later the question was taken up by the brothers Vicenzo Riccati and Giordano Riccati. They were sons of the celebrated Giacomo Francesco Riccati, of Venice, the originator of Riccati's differential equation. Vicenzo Riccati taught mathematics at the University of Bologna. In 1767 he addressed five letters to Jacopo Pellizzari, professor in Treviso, which, we are informed, were printed along with a letter of Giordano Riccati.[2] They were reprinted twelve years later.[3] The arguments of Leibniz, J. Bernoulli, D'Alembert, Euler and de Foncenex were discussed. As a whole, Riccati considers D'Alembert's criticisms of Euler valid, when Euler takes $1 + y/n = \cos(2\lambda - 1)/n\cdot\pi \pm \sqrt{-1}\sin(2\lambda - 1)/n\cdot\pi$, where y is the required $l(-1)$, and derives from it, for $n = \infty$, $l(-1) = \pm(2\lambda - 1)\pi\sqrt{-1}$, D'Alembert modifies the process by writing $\lambda = n$, $1 + y/n = \cos(2 - 1/n)\pi \pm \sqrt{-1}\sin(2 - 1/n)\pi$. Letting $1/n = 0$ in the right member, but not in the left, he gets $1 + y/n = 1$, or $y = 0 = l(-1)$. Riccati rejects D'Alembert's revision of Euler at this point, but sustains D'Alembert in criticizing Euler's $l(1 + \omega)^n = n\omega$, n infinite and ω infinitesimal, as embracing all the logarithms, since $(1 + \omega)^n$ must be a positive number (!), hence $n\omega$ does not by his analysis represent the logarithms of negative numbers. We have seen that Euler sets no limitation as to whether the infin-

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[1] Karsten, *op. cit.*, p. 103.

[2] P. Riccardi, *Biblioteca matematica Italiana*, Modena, 1893, Pt. I, Vol. II, 368.

[3] *Nuovo giornale de'letterati d'Italia*, Modena, 1779, T. XVI, pp. 137–219.

itesimal ω is real or imaginary and does not say that he means $(1+\omega)^n$ to represent a positive number. As a matter of fact, Euler puts later $(1+\omega)^n = x = -1$ and finds $l(-1)$ from this. Euler's lack of explicitness of statement at this point turned Riccati and others against him. Riccati writes $-1-\omega$ in place of $1+\omega$, thus making $l(-1-\omega)=\omega$, and arrives of course at $l(-1)=0$. Moreover, the logarithmic curve was found to have two branches. The view of J. Bernoulli I and D'Alembert prevailed. Vicenzo Riccati wrote a letter to Joachim Pessuti, who had been in St. Petersburg and was a friend of Euler. Pessuti was then publisher of two literary journals, the *Anthologia romana* and *Effemeridi letterarie,* and after 1787 professor of applied mathematics at the Collegio della Sapienza in Rome. Pessuti defended the views of Euler in a paper, entitled *Riflessioni analitiche,* Livorno, 1777,[1] deriving Euler's main results from $u = \int \frac{dz}{1+z^2} = \frac{1}{2\sqrt{-1}} \log \frac{1+z\sqrt{-1}}{1-z\sqrt{-1}}$. Putting $z=0$, he gets $u = n\pi$, where $n = 0, 1, 2, \cdots$. A reply to Pessuti made by V. Riccati appeared anonymously.[2] Riccati's attacks on Euler were repelled by a second writer who had come under the influence of St. Petersburg, the German astronomer Friedrich Theodor v. Schubert, in an article "Ueber die Logarithmen verneinter Grössen."[3] In the year 1778 Giuseppe Calandrelli, professor of mathematics at the Collegio Romano, attacked Pessuti and Euler in a pamphlet *Saggio analitico sopra la riduzione degli archi circolari ai logaritmi immaginarj,* Rome, 1778. As V. Riccati had done before him, so now Calandrelli casts doubts upon the correctness of the Bernoulli-Euler relation $\pi\sqrt{-1} = l(-1)$, the argument hinging on $2l(-1) = l1$. The same line of argument was pursued by P. M. Caldani, a pupil of V. Riccati and professor in Bologna. Caldani's *Della proporzione Bernoulliana fra il diametro, e la circonferenza del circolo e dei logaritmi,* Bologna, 1782, caused D'Alembert to pronounce Caldani the first mathematician in Italy.[4] Caldani objected, as Riccati had done before him, to Pessuti's writing $z\sqrt{-1}=0$, when $z=0$. Caldani writes $0 : 0\sqrt{-1} : : a : a\sqrt{-1}$; now, if $0 = 0\sqrt{-1}$, there follows $a = a\sqrt{-1}$, an impossibility. He wrote later a paper which we have not seen, *Riflessioni sopra un opuscolo del Padre Franceschini Barnabita, dei logaritmi de' numeri negativi,* Modena, 1791. These writers, including Pessuti, agree that Euler's proof of his fundamental theorem is not clear. When Euler writes $l(1+\omega)=\omega$, where ω is an infinitesimal, and then gets $l1 = 0$ when $\omega = 0$, he cannot at the same time claim, said the critics, that $\log 1 = 2\pi\sqrt{-1}, 4\pi\sqrt{-1}$, etc. Many of these points are discussed by the Spanish Jesuit Juan Andrès in a very interesting manner, in his book previously quoted. In 1766, when the

[1] Pessuti's paper is known to us only through the remarks on it found in Calandrelli's reply and in a book, *De studiis philosophicis et mathematicis,* Matriti, 1789, p. 215, written by JUAN ANDRÈS, and in FRÈRES (HOEFER), *Nouv. Biog. Générale.*

[2] *Nuovo giornale de'letterati d'Italia,* T. XV, Modena, 1778, pp. 144–204. See also T. XVIII, p. 107.

[3] Schubert's article is announced in the *Göttingische Anzeigen von gelehrten Sachen,* 1794, 2. Bd., p. 1313, as a reply to an article of Riccati in *Memorie della Soc. Italiana,* T. IV, p. 166.

[4] Frères, *op. cit.,* Art. *Caldani.*

Jesuits were driven out of Spain, he went to Italy and interested himself in the philosophic discussions there. He was inclined to accept Euler's conclusions, but thought that if $1 = e^0 = e^{2n\pi\sqrt{-1}}$, then we ought to have $0 = 2\pi\sqrt{-1} = 4\pi\sqrt{-1}$, etc.

Of higher type is the research of Gregorio Fontana, professor at the University of Pavia. He takes Euler's point of view. In an article, *Sopra i logaritmi delle quantità negativa e sopra gl'immaginarj*,[1] he tried to base the theory upon simpler and more rigorous proofs. He gives three proofs. First he takes $x = \text{vers}\ \phi$, and gets $\sqrt{-1}\, d\phi = \dfrac{dx}{\sqrt{x^2 - 2x}} = \left(\dfrac{xdx - dx}{\sqrt{x^2 - 2x}} + dx\right) \div (x - 1 + \sqrt{x^2 - 2x})$. Integrating, he gets $\phi\sqrt{-1} = \log\ (1 - x - \sqrt{x^2 - 2x})$ after letting $\phi = 0$ when $x = 0$. For $x = 2$, $\log\ (-1) = \pm (2n - 1)\pi\sqrt{-1}$; for $x = 0$, $\log 1 = \pm 2n\pi\sqrt{-1}$. A second proof starts from $x = \cos \phi$ and involves integration. A similar third proof implicates several trigonometric functions. Fontana reasons with a clearness which is unusual in articles on logarithms of that day. He tells how at one time he thought he had absolute proof of the falsity of Euler's results. In $e^{\log(-a)} = -a$ he thought that $\log\ (-a)$ must surely be real whenever $-a$ is real. As a matter of fact, says he, we have $\log\ (-a) = \pm (2n - 1)\pi\sqrt{-1} + \log a$. Finally Fontana derives the theorems without using the integral calculus; he uses infinite series, but pays no attention to questions of convergence.

This able article did not prevent the appearance of five "proofs" that $\log(-z) = \log z$, in Pietro Franchini's *Teoria dell'Analisi*, Rome, 1792. In 1795 Gianfrancesco Malfatti, of the University of Ferrara, discussed at length the question whether the logarithmic curve has one or two branches.[2] The equation $ydx = dy$ has an infinite number of integrals even under the restriction that $x = 0$ when $y = 1$. For any constant n, $e^{nx} = y^n$ gives rise to the differential equation $ydx = dy$. Before integrating one must agree upon the number of values y shall have. If it is to have one value, we get $e^x = y$; if two values, we get $e^{2x} = y^2$ or $(e^x - y)(e^x + y) = 0$. That $e^x = y$ has only one branch is argued from geometric and analytic considerations.

In regular mathematical text-books the lack of agreement relating to $\log\ (-1)$ was as marked as in the special articles. Odoardo Gherli [3]takes the ground that negative numbers differ from the positive simply in being taken in a contrary sense, and that logarithms take no cognizance of such contrarity. Pietro Paoli[4]

[1] *Memorie di matematica e fisica della società italiana*, T. I, Verona, 1782, pp. 183–202.

[2] *Memorie della reale accademia di scienze . . . di Mantova*, T. I, pp. 2–54. This article is known to us through a valuable historical monograph by Bernard F. Thibaut, *Dissertatio historiam controversiæ circa numerorum negativorum et impossibilium logarithmos sistens*, Göttingen, 1797, p. 20. Malfatti's article is reviewed in the *Göttingische Anzeigen von gelehrten Sachen*, 1796, 2. Bd., p. 1242. The *Göttingische Anzeigen* for 1786, 2. Bd., p. 1029, refers to another eighteenth century historical paper on this subject, namely the *De logarithmis numerorum negativorum* by Fridrich Mallet, professor at Upsala, which was printed in the *Nova Acta Reg. Soc. Scient. Upsaliensis*, Vol. IV, 1784, pp. 205–220.

[3] *Gli elementi teorico-pratici*, T. II, Modena, 1771, p. 297.

[4] *Elementi d'Algebra*, T. I, Pisa, 1794.

derives "il famoso Teoreme del Sig. Euler," but thinks he must use infinite series to secure rigor. Paolo Frisi, in his algebra,[3] shows familiarity with the leading writers on logarithms; he states the results reached by Euler, but is out of sympathy with them. Many results appear to him absurd, as for instance, $\log(-1) : \sqrt{-1} = \pi : 1$. He objects also to $0 \cdot \sqrt{-1} = 0$ and claims that an imaginary zero represents a real magnitude different from zero.

In a ponderous quarto volume,[2] published in 1782 at Florence, Petro Ferroni devotes a chapter to logarithms of negative numbers, only to arrive at the conclusion that analytic, as well as geometric, considerations of continuity yield $l(+1) = l(-1)$. Euler's argument, $-y = y(-1) = c^{ly} \cdot c^{\pm\lambda p\sqrt{-1}}$, hence $l(-y) = ly \pm \lambda p\sqrt{-1}$, is rejected, since by a purer doctrine (*puriore religione*) one has $-y = -c^{ly}$, and $l(-y)$ a real number. Ferroni bases his theory of general logarithms on the relation $y = \pm c^x$; he perceives no inconsistency in this procedure.

Signs of decadence are visible also in Germany. The high level reached there in the middle of the century was not maintained at its close. Kästner,[3] of the University of Göttingen, shows from the theory of ratios that we cannot have real logarithms of negative quantities. I. A. C. Michelsen, an admirer of Euler, a teacher at the royal gymnasium in Berlin, who in 1788 brought out a German translation of Euler's famous *Introductio* of 1748, added notes to that translation which indicate that Michelsen failed completely to grasp Euler's[4] argument; he holds to the view, for instance, that to every number, whether positive or negative, there corresponds one and only one logarithm. In 1795, G. S. Klügel[5] declared himself as follows: "The $\log(-x^2)$ is impossible; but the $\log(-x)$ is possible, else it must be shown that some condition does not permit x to be taken negatively." The possibility of the logarithm of a negative number rests upon the possibility of the negative number itself. Interest in the discussion of logarithms is shown in occasional notices that appeared in the *Göttingische Anzeigen von gelehrten Sachen.*[6] A notice of the year 1786 declares that the opponents of Euler tried to decide the question by formulæ of integration, radii of curvature, etc., thereby endeavoring to pass upon fundamental concepts by means of systems of notation, when, as a matter of fact, the concepts must be determined upon first, before the systems of notation are brought into action. Euler's opponents are dubbed calculators, rather than philosophers.

In conclusion, we stop to inquire, What was the result of the half-century of discussion?

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[1] *Paulli Frisii Operum tomus primus*, Mediolani, 1782, p. 188, etc.

[2] P. Ferroni, *Magnitudinum exponentialium logarithmorum et trigonometriæ sublimis theoria nova methodo pertractata.* Florentiæ, 1782, Chap. IX.

[3] *Leipziger Magazine f. reine u. angew. Math.*, Stück IV, 1786, p. 531. See also Cantor, *op. cit.*, IV, 1908, p. 313; B. F. Thibaut, *op. cit.*, p. 21.

[4] Euler, *Einleitung in d. Analysis d. Unendlichen*, 1788, p. 503.

[5] *Hindenburg Archiv*, 1795, 3. Heft, pp. 309–319; 4. Heft, pp. 470–481.

[6] See 2. Bd., 1786, p. 1029; 2. Bd., 1792, pp. 1185, 1186; 2. Bd., 1794, pp. 1029, 1313; 2. Bd., 1796, p. 1242.

The prevailing feeling toward the theory of logarithms of negative numbers was one of hesitancy. In 1799 Christian Kramp, professor of mathematics in Strassburg, expressed doubts[1] on the correctness of the formula $l(-x) = l(-1) + lx$.

In 1801 Robert Woodhouse, of Caius College, published a paper "On the Necessary Truth of Certain Conclusions obtained by Means of Imaginary Quantities." He says:[2] "The paradoxes and contradictions mutually alleged against each other, by mathematicians engaged in the controversy concerning the application of logarithms to negative and impossible quantities, may be employed as arguments against the use of those quantities in investigation."

In 1803 L. N. M. Carnot said in his *Géométrie de position*, p. III, that the long debate has failed to clear up the paradox that, while $\log(-2)^2 = \log(2)^2$, we cannot write $2\log(-2) = 2\log 2$. Euler had this point cleared up in 1749; he had the assumptions, under which $2\log(-2)$ is or is not equal to $2\log 2$, set forth in masterly fashion, but the discussion of fifty years involved the more primitive question, whether $\log x$ is or is not equal to $\log(-x)$. Why was Euler's brilliant paper of 1749 not convincing? For three reasons:

1. Euler failed to make plain that the question, whether the logarithmic curve had one branch or two branches, had in reality nothing to do with the theory of logarithms of negative and complex numbers.
2. Euler's proof of his fundamental theorem, that $\log x$ has an infinite number of values, was not explained with sufficient fulness of detail to carry general conviction.
3. The rank and file of mathematicians had not yet learned to treat with care the many-valued inverse operations and the vexing questions of discontinuity. Nor did they avoid the danger of confusion resulting from the simultaneous consideration of logarithms of different bases.

A noteworthy feature of the long discussion deserves mention. Ordinarily one genuine proof is accepted as sufficient to establish a mathematical truth with absolute certainty; no argument can be advanced against such a proof. But, as a rule, the discussion of logarithms of complex numbers was considered as involving arguments on each side of the question, so that the decision seemed to rest on a preponderance of arguments or a preponderance of probabilities, which apparently rendered it desirable for a partisan, in lawyer's fashion, to advance as many different "proofs" as possible. This attitude was particularly noticeable in the debates carried on by correspondence. It was due mainly to a lack of sharp definition of terms and to a failure to discriminate between what is assumed without proof and what is to be subjected to rigorous demonstration.

The Union of the Logarithmic and Exponential Concepts.

The wider adoption of the definition of logarithms as exponents, in school books, was due largely to the influence and example of L. Euler, who gave it in

[1] Montferrier's *Dic. d. scienc. math.*, *T.* II, Paris, 1836, p. 185, Art. "Logarithme," where reference is made to Kramp's *Analyse des réfractions astronomiques et terrestres*, Strasbourg, 1799.

[2] *Philosoph. Transactions*, 1801, p. 111.

his *Anleitung zur Algebra.*[1] During the eighteenth century the Naperian definition, based on the two progressions, continued to be the prevailing definition. As late as 1808, C. F. Kaussler expressed decided preference for it, the new definition offering to beginners "gaps and obscurities."[2]

We have seen that Euler extended the exponential concept even to the use of imaginary exponents. We have seen that the expression $\sqrt{-1}^{\sqrt{-1}}$ is associated with his name; he recognized that it was infinitely many-valued. When the exponent is real and fractional, the multiple values of $a^{m/n}$ were recognized by Euler and others in research articles. As a rule, this many-valuedness was not discussed at all in text-books of the eighteenth century, such as the algebras of Saunderson, Blassière and Lacroix, or the mathematical texts of La Caille, J. F. Häseler, Abbé Sauri, Karsten, Reyneau, Bézout, Gherli, Rivard and Bertrand, or the mathematical dictionaries of E. Stone and Ch. Wolf.

The relation $a = e^{\log a}$ was easily deduced from the new definition of logarithms and was used by some writers of this century, such as G. Fontana in 1782.

[Further instalments will follow, carrying this history through the nineteenth century. EDITORS.]

SOME INVERSE PROBLEMS IN THE CALCULUS OF VARIATIONS.

By E. J. MILES, Yale University.

In the so-called inverse problem[1] of the calculus of variations a doubly infinite family of curves

$$y = f(x, \alpha, \beta) \tag{1}$$

is given and it is required to find a function $F(x, y, y')$ such that the given system of curves forms the system of extremals for the integral

$$I = \int_{x_0}^{x_1} F(x, y, y')dx. \tag{2}$$

Darboux[2] has shown that the problem always has an infinite number of solutions which can be reduced to quadratures. For if in the Euler differential equation

$$F_y - F_{y'x} - y'F_{y'y} - y''F_{y'y'} = 0$$

the expression $G(x, y, y')$, where $y'' = G(x, y, y')$ is the differential equation

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[1] L. Euler, *Anleitung zur Algebra*, St. Petersburg, 1770, 1. Theil, Cap. 21.

[2] C. F. Kaussler, *Lehre von den Logarithmen*, Tübingen, 1808, preface. Quoted by Tropfke Gesch. d. Elem.-Math., 2. Band, Leipzig, 1903, p. 142.

[1] For a treatment of this problem see for instance Bolza: *Vorlesungen über Variationsrechnung*, § 6c.

[2] Darboux: *Theorie des Surfaces*, vol. III, Nos. 604, 605.